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Real numbers in the topos of sheaves over the category of filters

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Abstract

The sheaves over the category of filters, with the precanonical topology, serve as a universe of sets where nonstandard analysis can be developed along constructive principles. In this paper we show that the Dedekind real numbers of this topos can be characterised as the nonstandard hull of the rational numbers. Moreover, it is proved that the axiom of choice holds on standard sets of the topos. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In nonstandard analysis [11] the real numbers can be constructed as a nonstandard hull of the rationals,

$$\mathbb{R} \cong \mathrm{Fin}({}^*\mathbb{Q})/I. \tag{1}$$

Here ${}^*\mathbb{Q}$ is a nonstandard extension (e.g., a suitable ultrapower) of the rational numbers, $\mathrm{Fin}({}^*\mathbb{Q})$ denotes the subring of finite or bounded elements of ${}^*\mathbb{Q}$ and I is the maximal ideal of infinitesimals in this ring. The nonstandard hull of $\mathbb{Q}$ is prototypical for many constructions in nonstandard analysis that exploit hyperfinite approximations to uncountable standard objects.

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Moerdijk [6] introduced a sheaf-theoretic method for making nonstandard extensions analogous to those given by ultrapowers. This made it possible to develop nonstandard arithmetic [6] and nonstandard analysis [7–9] constructively, while keeping many of the formal properties of the classical setting. Instead of regarding the nonstandard extension ${}^*A$ of a set A as a reduced power, i.e., as the set of sequences from A defined on a particular (ultra-)filter, Moerdijk defined it as the representable sheaf $\mathrm{Hom}(-,A)$ on a category of filters. For such a category of filters $\mathbb{B}$, equipped with the precanonical topology (see Section 2), the topos of sheaves $\mathscr{N} = \mathrm{Sh}(\mathbb{B})$ is a universe of sets and nonstandard sets.

In the present paper we show that (1) holds internally to $\mathscr{N}$ when $\mathbb{R}$ is the set of Dedekind reals (see Section 4). We also obtain a constructive version of the standard part map (cf. [11]). As preparation we show that the topos $\mathscr{N}$ validates dependent choice (Section 3), and hence that Cauchy reals and Dedekind reals coincide in this topos. Only constructive methods complying with Bishop constructivism [1] are used, except when verifying the full axiom of choice for standard sets (Lemma 3.1 and Theorem 3.2) and in Proposition 4.5(ii). Recall that Bishop constructivism admits the use of dependent choice, which can be justified directly from the constructive reading of the logical connectives. Naturally, all results are valid in classical set theory.

2. Sheaves over the category of filters

The category of filters has been studied extensively by Blass [2] and Koubek and Reiterman [4]. We consider here the equivalent category $\mathbb{B}$ of filter bases, which is better adapted to a constructive treatment. In this section, we briefly recall some basic definitions and facts from [6–8]. (Remark 2.2 is new however.)

A *filter base* $\mathscr{F} = (F, \{F_i\}_{i\in I})$ consists of an inhabited index set I (i.e., a set containing at least one element) and a family $\{F_i\}_{i\in I}$, called the *base sets*, consisting of subsets of the *underlying set* F and satisfying the filtering condition: for each pair $i, j \in I$, there exists $k \in I$ such that $F_k \subseteq F_i \cap F_j$. The filter generated is then $\{S \subseteq F\colon (\exists i \in I)\, F_i \subseteq S\}$. For constructive reasons, it will be better to work only with the bases of filters. In the sequel we shall abuse the language and simply call them *filters*. A filter $\mathscr{F} = (F, \{F_i\}_{i\in I})$ is *proper* if every base set is inhabited. A proper filter $\mathscr{F}$ is an *ultrafilter* if for every $S \subseteq F$ either $S \supseteq F_i$ or $F \setminus S \supseteq F_i$ for some $i \in I$.

Let $\mathscr{F} = (F, \{F_i\}_{i\in I})$ and $\mathscr{G} = (G, \{G_j\}_{j\in J})$ be filters. A *continuous map* α from $\mathscr{F}$ to $\mathscr{G}$, in symbols $\alpha : \mathscr{F} \to \mathscr{G}$, is a partial function $\alpha : F \to G$ which is totally defined on some base set of $\mathscr{F}$, and satisfies the *continuity* condition, $(\forall j \in J)(\exists i \in I)\, \alpha[F_i] \subseteq G_j$. Two such morphisms are equivalent if they agree on some base set of $\mathscr{F}$. The filters together with the continuous maps then form a category $\mathbb{B}$ with terminal object and all pullbacks, see [6,7]. For each set A there is a *trivial filter* $\bar{A} = (A, \{A\})$; a morphism $\alpha : \bar{A} \to \bar{B}$ is simply a function $A \to B$. Thus, the category of sets can be considered as a (full) subcategory $\mathbb{S}$ of $\mathbb{B}$. We now define a Grothendieck topology K on $\mathbb{B}$. Let $\mathscr{G}^{(k)} = (G^{(k)}, \{G_j^{(k)}\}_{j\in J^{(k)}})$, $k = 1, \ldots, n$, and $\mathscr{F} = (F, \{F_i\}_{i\in I})$ be filters. A finitely enumerable set of continuous maps $\{\beta_k : \mathscr{G}^{(k)} \to \mathscr{F}\}_{k=1}^n$ is called a *K-cover* of $\mathscr{F}$ if for all $j_1 \in J^{(1)}, \dots, j_n \in J^{(n)}$ there exists $i \in I$ for which

$$\beta_1[G_{j_1}^{(1)}] \cup \cdots \cup \beta_n[G_{j_n}^{(n)}] \supseteq F_i. \tag{2}$$

It can be shown [4] that a map $\alpha : \mathscr{F} \to \mathscr{G}$ makes up a cover $\{\alpha\}$ iff α is epi. An epimorphism $\alpha : \mathscr{F} \to \mathscr{G}$ is usually written $\alpha : \mathscr{F} \twoheadrightarrow \mathscr{G}$. Thus, using [6, p. 40], a family $\{\beta_k : \mathscr{G}^{(k)} \to \mathscr{F}\}_{k=1}^n$ is a cover iff the canonical map $\mathscr{G}^{(1)} + \cdots + \mathscr{G}^{(n)} \to \mathscr{F}$ is epi. Also, covers are preserved by pullbacks. Thus, K is the *precanonical* topology on $\mathbb{B}$ (cf. [3]).

Note that the set of morphisms from $\mathscr{F}$ to $\bar{A}$, $\mathrm{Hom}_{\mathbb{B}}(\mathscr{F}, \bar{A})$, can be identified with the reduced power $A^I/\mathscr{F}$. The *nonstandard extension* of A is the representable sheaf ${}^*A =_{\mathrm{def}} \mathbf{y}(\bar{A}) = \mathrm{Hom}_{\mathbb{B}}(-, \bar{A}) : \mathbb{B}^{\mathrm{op}} \to \mathit{Sets}$. If R is a subset of A, then ${}^*R$ is a subobject of ${}^*A$, since the Yoneda embedding is left exact. Let $\mathscr{S} = \mathit{Sets}$ and $\mathscr{N} = \mathrm{Sh}(\mathbb{B})$ and let $L_{\mathscr{S}}$ and $L_{\mathscr{N}}$ be their respective internal languages. For any $L_{\mathscr{S}}$-formula Φ, we define its *-transform, ${}^*\Phi$, to be the $L_{\mathscr{N}}$-formula where all symbols have been replaced by their starred counterparts. Thus, for instance

$$(\exists y \in {}^*M)(\forall x \in {}^*S)\, {}^*f(x)\, {}^*\!< y$$

is the *-transform of $(\exists y \in M)(\forall x \in S) f(x) < y$. The locally constant sheaf $\Delta(A)$ for a set A can be defined as a subsheaf of ${}^*A$ in the following manner. For $\varphi \in {}^*A(\mathscr{F})$ let $\varphi \in \Delta(A)(\mathscr{F})$ iff for some base set F_i, the image $\varphi[F_i]$ is included in some $\{a_1, \dots, a_n\} \subseteq A$. The sheaf $\Delta(A)$ comprises the *standard elements* of ${}^*A$. For any $a \in A$, the constant function ${}^*a \in {}^*A(\mathscr{F})$ is clearly in $\Delta(A)(\mathscr{F})$. A formula which is a *-transform of an $L_{\mathscr{S}}$-formula is called *internal.* The following theorem is due to Moerdijk (see [6,7]):

Theorem 2.1 (Moerdijk [6]). *Let $\Phi(x_1^{S_1}, \dots, x_n^{S_n})$ be an $L_{\mathscr{S}}$-formula, and suppose that $\alpha_1 \in {}^*S_1(\mathscr{F}), \dots, \alpha_n \in {}^*S_n(\mathscr{F})$. Then for $\bar{\alpha} = \alpha_1, \dots, \alpha_n$,*

$$\mathscr{F} \Vdash {}^*\Phi(\alpha_1, \dots, \alpha_n) \quad \textit{iff} \quad (\exists i \in I)(\forall u \in F_i)\, \Phi(\alpha_1(u), \dots, \alpha_n(u)).$$

As a direct consequence of the theorem we have the *transfer principle*, that for any $L_{\mathscr{S}}$-formula Φ: ${}^*\Phi$ holds in $\mathscr{N}$ iff Φ is true. See [6–8] for other useful principles of nonstandard analysis: idealisation, overspill and underspill.

Remark 2.2. (i) The proof in [4] that $\alpha : \mathscr{F} \to \mathscr{G}$ is epi iff $\{\alpha\}$ is a K-cover, is nonconstructive. However, it can be constructivised as follows. The direction ($\Leftarrow$) is immediate, whereas to prove ($\Rightarrow$) we use a standard trick. Suppose that α is epi and consider an arbitrary F_i within the domain of definition for α. Consider the trivial filter $\mathscr{H} = \overline{\wp(\{0\})}$ and define $\beta, \gamma : \mathscr{G} \to \mathscr{H}$ by $\beta(x) = \{0 : (\exists u \in F_i)\alpha(u) = x\}$ and $\gamma(x) = \{0\}$. For $u \in F_i$, we have $\beta(\alpha(u)) = \gamma(\alpha(u))$. Since α is epi, $\beta = \gamma$. Thus, for some base set G_j of $\mathscr{G}$: $\forall x \in G_j\, \beta(x) = \gamma(x)$, i.e., $G_j \subseteq \alpha[F_i]$.

(ii) Given that the topology K is precanonical, it is natural to ask whether $\mathbb{B}$ is a pretopos; see [3]. By refining a counterexample due to Blass [2], we can show that

$\mathbb{B}$ lacks coequalizers even for equivalence relations, thus giving a negative answer. Let $\mathscr{R} = (\mathbb{N} \times \mathbb{N}, \{R_n\}_{n \in \mathbb{N}})$ be the filter given by $R_n = \{(x, y) \in \mathbb{N} \times \mathbb{N}: x = y \text{ or } x \geq n \text{ and } y \geq n\}$. The first and second projections $\pi_1, \pi_2 : \mathscr{R} \to \bar{\mathbb{N}}$ are continuous and this pair of maps constitutes an equivalence relation. Assuming that this pair has a coequalizer leads to a contradiction by following the same line of reasoning as in [2, Example 11].

3. The axiom of choice

We show in this section that dependent choice is valid in $\mathscr{N}$ for the standard natural numbers $\mathbf{N} = \Delta(\mathbb{N})$. If we assume a classical metatheory, such as ZFC, the axiom of choice holds on any standard set $\Delta(S)$ in $\mathscr{N}$. The latter result depends on the following lemma. Let $S^{<\omega}$ be the set of finite sequences of members of S; let $\sigma * \tau$ denote the concatenation of the sequences σ and τ. We write $\sigma \leq \tau$, if there are sequences ρ and ρ' such that $\tau = \rho * \sigma * \rho'$.

Lemma 3.1. *Suppose that $\{\psi_x : \mathscr{F}_x \to \mathscr{F}\}_{x \in S}$ is a family of epimorphisms. Then there exists a filter $\mathscr{G}$ and a family of epimorphisms $\{\gamma_x : \mathscr{G} \to \mathscr{F}_x\}_{x \in S}$ such that $\psi_x \gamma_x = \psi_y \gamma_y$ for all $x, y \in S$.*

Proof. For each sequence $\sigma = (x_1, \ldots, x_n) \in S^{<\omega}$ let $\mathscr{F}_\sigma$ be the pullback

$$\mathscr{F}_{x_1} \times_{\mathscr{F}} \cdots \times_{\mathscr{F}} \mathscr{F}_{x_n} \tag{3}$$

of the maps $\psi_{x_i} : \mathscr{F}_{x_i} \to \mathscr{F}$, $i = 1, \ldots, n$. Denote by $\pi_i^\sigma : \mathscr{F}_\sigma \to \mathscr{F}_{x_i}$ the ith projection from this pullback. Each map π_i^σ is epi, since epis are preserved by pullbacks. Note that $\mathscr{F}_{()}$ is just $\mathscr{F}$ and $\mathscr{F}_{(x)}$ is $\mathscr{F}_x$.

For any subsequence $\sigma = (x_i, x_{i+1}, \ldots, x_{i+j})$ of $\tau = (x_1, x_2, \ldots, x_n)$, there exists, by the pullback property, a unique map $\varphi_{\tau,\sigma} : \mathscr{F}_\tau \to \mathscr{F}_\sigma$ such that $\pi_k^\sigma \varphi_{\tau,\sigma} = \pi_{i-1+k}^\tau$ for $k = 1, \ldots, j+1$. For a filter $\mathscr{F}_\sigma$ write its base sets as $F_{\sigma,p}, F_{\sigma,q}, \ldots$. Let $F_{(),p_{()}}$ be any base set of $\mathscr{F}$. Then, define a family of base sets $\{F_{\tau,p_\tau}\}_{\tau \in S^{<\omega}}$ by induction on the length of τ, in such a way that for all $\tau, \sigma \in S^{<\omega}$ with $\tau \geq \sigma$

$$\varphi_{\tau,\sigma}[F_{\tau,p_\tau}] \subseteq F_{\sigma,p_\sigma}$$

and $\varphi_{\tau,\sigma}$ is defined on F_{τ,p_τ}. Construct a filter base $\mathscr{G}$ with underlying set

$$G = \{(\sigma, x) \colon \sigma \in S^{<\omega},\ x \in F_{\sigma,p_\sigma}\}$$

and base sets

$$G_{(\sigma,q)} = \{(\tau, y) \in G \colon \tau \geq \sigma \text{ and } \varphi_{\tau,\sigma}(y) \in F_{\sigma,q}\}$$

for (σ, q) such that $F_{\sigma,q} \subseteq F_{\sigma,p_\sigma}$. We check that this is indeed a filter base. Suppose that (σ, q) and (σ', q') are given. Let $\tau = \sigma * \sigma'$. Take q'' to be such that $F_{\tau,q''} \subseteq F_{\tau,p_\tau}$, $\varphi_{\tau,\sigma}[F_{\tau,q''}] \subseteq F_{\sigma,q}$ and $\varphi_{\tau,\sigma'}[F_{\tau,q''}] \subseteq F_{\sigma',q'}$. For $(\tau', y) \in G_{(\tau,q'')}$, we have

$\varphi_{\tau',\sigma}(y) = \varphi_{\tau,\sigma}(\varphi_{\tau',\tau}(y)) \in \varphi_{\tau,\sigma}[F_{\tau,q''}] \subseteq F_{\sigma,q}$. Hence, $G_{(\tau',q'')} \subseteq G_{(\sigma,q)}$. Similarly, one shows $G_{(\tau',q'')} \subseteq G_{(\sigma',q')}$. Thus, $\mathscr{G}$ is a filter base.

For each $x \in S$ define $\gamma_x : \mathscr{G} \to \mathscr{F}_{(x)} = \mathscr{F}_x$ on $G_{((x),q_{(x)})}$ by $\gamma_x(\tau, y) = \varphi_{\tau,(x)}(y)$. Then γ_x is continuous since for each p there exists some p' with $\varphi_{\tau,(x)}[F_{\tau,p'}] \subseteq F_{(x),p}$ and $F_{\tau,p'} \subseteq F_{\tau,p_\tau}$. It follows that $\gamma_x[G_{(\tau,p')}] \subseteq F_{(x),p}$, proving that γ_x is continuous. To see that γ_x is epi, note that for any $G_{(\sigma,q)} \subseteq G_{((x),p_{(x)})}$, with $\sigma \geq (x)$ and $F_{\sigma,q} \subseteq F_{\sigma,p_\sigma}$, we have $\gamma_x[G_{(\sigma,q)}] \supseteq F_{\sigma,q}$.

Now let $x, y \in S$. We shall prove that $\psi_x\gamma_x = \psi_y\gamma_y$. Let ψ_x and ψ_y be defined on $F_{(x),p}$ and $F_{(y),p'}$, respectively. Let $G' = G_{(\sigma,q)}$ be a base set within the domain of definition of both γ_x and γ_y and such that $\sigma \geq (y)$, $\gamma_x[G'] \subseteq F_{(x),p}$ and $\gamma_y[G'] \subseteq F_{(y),p'}$. Then for $(\tau,z) \in G'$,

$$\psi_x(\gamma_x(\tau,z)) = \psi_x\varphi_{\tau,(x)}(z) = \psi_x\varphi_{\sigma,(x)}\varphi_{\tau,\sigma}(z) = \psi_y\varphi_{\sigma,(y)}\varphi_{\tau,\sigma}(z) = \psi_y\gamma_y(\tau,z).$$

The third equation is immediate from the pullback diagram for (3). □

The following gives the axiom of choice on any inhabited standard set.

Theorem 3.2. *Let S be an inhabited set, and put $\mathbf{S} = \Delta(S)$. Let T and P be arbitrary sheaves in $\mathscr{N}$. For any subobject $R \subseteq \mathbf{S} \times T \times P$ the following is valid in the internal logic of $\mathscr{N}$:*

$$\forall z\,[\forall x\ \exists y\ R(x,y,z) \to (\exists f \in T^{\mathbf{S}})\,\forall x\ R(x,f(x),z)].$$

Proof. Suppose $\mathscr{F} \Vdash (\forall x \in \mathbf{S})\,\exists y\ R(x,y,\zeta)$ where $\zeta \in P(\mathscr{F})$. Hence, for each $x \in S$ there exists an epi $\varphi_x : \mathscr{F}_x \twoheadrightarrow \mathscr{F}$ and $y_x \in T(\mathscr{F}_x)$ such that

$$\mathscr{F}_x \Vdash R({}^*x, y_x, \zeta\varphi_x).$$

By Lemma 3.1 there exist a filter $\mathscr{G}$ and a family of epimorphisms $\{\gamma_x : \mathscr{G} \to \mathscr{F}_x\}$ such that $\varphi_x\gamma_x = \varphi_y\gamma_y$ for all $x, y \in S$. Let $x_0 \in S$ and put $\varepsilon = \varphi_{x_0}\gamma_{x_0}$. Hence, for each $x \in S$

$$\mathscr{G} \Vdash R({}^*x, y_x\gamma_x, \zeta\varepsilon).$$

Let $[\mathbf{S} \to T]$ be the exponent of $\mathbf{S}$ and T in $\mathscr{N}$. We find $f \in [\mathbf{S} \to T](\mathscr{G})$ such that $\mathrm{ev}_{\mathscr{G}}(f, {}^*x) = y_x\gamma_x$. Thus,

$$\mathscr{G} \Vdash R({}^*x, \mathrm{ev}(f, {}^*x), \zeta\varepsilon)$$

for each $x \in S$. It follows that

$$\mathscr{G} \Vdash (\exists f \in [\mathbf{S} \to T])(\forall x \in \mathbf{S})\,R(x, \mathrm{ev}(f,x), \zeta\varepsilon).$$

Since ε is epi, this last statement is forced already at $\mathscr{F}$. □

Next we show that dependent choice holds in $\mathscr{N}$. First we need an auxiliary result analogous to Lemma 3.1.

Lemma 3.3. *Suppose that* $\varphi_n : \mathscr{F}_{n+1} \to \mathscr{F}_n$, $n = 0, 1, 2, \ldots,$ *is a sequence of epimorphisms in* $\mathbb{B}$. *Then for some* $\mathscr{G}$ *there exists a sequence of epimorphisms* $\gamma_n : \mathscr{G} \to \mathscr{F}_n$ *such that* $\varphi_n \gamma_{n+1} = \gamma_n$ *for all* $n = 0, 1, 2, \ldots$.

Proof. Let F_{0,p_0} be any base set of $\mathscr{F}_0$. Define inductively a sequence p_n such that for each $n \in \mathbb{N}$, φ_n is defined on $F_{n+1,p_{n+1}}$ and $\varphi_n[F_{n+1,p_{n+1}}] \subseteq F_{n,p_n}$. For $m \geq n$ let $\varphi_{m,n} : \mathscr{F}_m \to \mathscr{F}_n$ be the composite $\varphi_n \varphi_{n+1} \cdots \varphi_{m-1}$.

Let $\mathscr{G}$ be the filter base with underlying set $\{(n,x)\colon n \in \mathbb{N} \text{ and } x \in F_{n,p_n}\}$ and whose base sets are

$$G_{(m,p)} = \{(n,x)\colon n \geq m \text{ and } \varphi_{n,m}(x) \in F_{m,p}\}$$

for pairs (m,p) such that $F_{m,p} \subseteq F_{m,p_m}$. As in Lemma 3.1, this is shown to be a filter base. The epimorphism $\gamma_n : \mathscr{G} \to \mathscr{F}_n$ is defined on G_{n,p_n} by $\gamma_n(m,x) = \varphi_{m,n}(x)$. Then clearly $\varphi_n \gamma_{n+1} = \gamma_n$ for all n. □

The following is then the statement that dependent choice holds in $\mathscr{N}$. It implies countable choice.

Theorem 3.4. *Let* A *and* P *be sheaves in* $\mathscr{N}$. *For any subobject* $R \subseteq \mathbf{N} \times A \times A \times P$ *the following is valid in the internal logic of* $\mathscr{N}$:

$$\forall z\, [\forall n\, \forall x\, \exists y\, R(n,x,y,z) \to \forall x (\exists f \in A^{\mathbf{N}})(f(0) = x \wedge \forall n\, R(n, f(n), f(n+1), z))].$$

Proof. Suppose that for some $\zeta \in P(\mathscr{F})$,

$$\mathscr{F} \Vdash \forall n\, \forall x\, \exists y\, R(n,x,y,\zeta). \tag{4}$$

Given $\beta : \mathscr{G} \to \mathscr{F}$ and $\xi \in A(\mathscr{G})$, let $\mathscr{G}_0 = \mathscr{G}$ and $\xi_0 = \xi$. Suppose that we have constructed sequences of epimorphisms

$$\mathscr{G}_n \overset{\gamma_{n-1}}{\twoheadrightarrow} \mathscr{G}_{n-1} \overset{\gamma_{n-2}}{\twoheadrightarrow} \cdots \twoheadrightarrow \mathscr{G}_1 \overset{\gamma_0}{\twoheadrightarrow} \mathscr{G}_0 \tag{5}$$

and $\xi_n \in A(\mathscr{G}_n), \ldots, \xi_1 \in A(\mathscr{G}_1)$ such that

$$\mathscr{G}_{k+1} \Vdash R(k, \xi_k \gamma_k, \xi_{k+1}, \zeta\beta\gamma_0 \cdots \gamma_k) \tag{6}$$

for $k = 0, \ldots, n-1$. From (4) follows $\mathscr{G}_n \Vdash \forall n\, \forall x\, \exists y\, R(n,x,y,\zeta\beta\gamma_0 \cdots \gamma_{n-1})$. Inserting ξ_n for x, we get $\gamma_n : \mathscr{G}_{n+1} \twoheadrightarrow \mathscr{G}_n$ and $\xi_{n+1} \in A(\mathscr{G}_{n+1})$ such that $\mathscr{G}_{n+1} \Vdash R(n, \xi_n\gamma_n, \xi_{n+1}, \zeta\beta\gamma_0 \cdots \gamma_{n-1}\gamma_n)$. Thus we have prolonged the sequences (5) and (6) by one unit each.

By Lemma 3.3 there exists $\eta_n : \mathscr{H} \twoheadrightarrow \mathscr{G}_n$, $n = 0, 1, 2, \ldots,$ such that $\gamma_n \eta_{n+1} = \eta_n$. It follows that

$$\mathscr{H} \Vdash R(n, \xi_n\eta_n, \xi_{n+1}\eta_{n+1}, \zeta\beta\eta_0).$$

Let $f \in [\mathbf{N} \to A](\mathscr{H})$ be such that $\mathrm{ev}_{\mathscr{H}}(f, {}^*n) = \xi_n \eta_n$. Consequently,

$$\mathscr{H} \Vdash \exists f\, (f(0) = \xi_0\eta_0 \wedge \forall n\, R(n, f(n), f(n+1), \zeta\beta\eta_0)).$$

Now since η_0 is epi, and both ξ_0 and β are arbitrary

$$\mathscr{F} \Vdash \forall x\, \exists f\, (f(0) = x \wedge \forall n\, R(n, f(n), f(n+1), \zeta)). \qquad \square$$

Remark 3.5. There is also a smaller "function space" $\Delta(\mathbb{N} \to \mathbb{N})$ for which countable choice is not valid; see [8, Proposition 6].

4. Dedekind reals and the standard part map

In this section we establish in the internal logic of $\mathcal{N}$ the isomorphism

$$\mathbb{R}_d \cong \mathrm{Fin}({}^*\mathbb{Q})/I, \tag{7}$$

where $\mathbb{R}_d$ is the set of Dedekind real numbers. First, we recall some facts and definitions from [3,5,8]. The sheaf of rational numbers in $\mathcal{N}$ is $\mathbf{Q} = \Delta(\mathbb{Q})$. A *Dedekind cut* is a pair of subobjects $L \subseteq \mathbf{Q}$ and $R \subseteq \mathbf{Q}$ satisfying (D1)–(D5) below.

(D1) $(\exists q \in \mathbf{Q})\, q \in L$, $(\exists r \in \mathbf{Q})\, r \in R$.
(D2) $(\forall q, r \in \mathbf{Q})[q < r \wedge r \in L \to q \in L]$, $(\forall q, r \in \mathbf{Q})[q < r \wedge q \in R \to r \in R]$.
(D3) $(\forall q \in \mathbf{Q})[q \in L \to (\exists r \in \mathbf{Q})(q < r \wedge r \in L)]$, $(\forall r \in \mathbf{Q})[r \in R \to (\exists q \in \mathbf{Q})(q < r \wedge q \in R)]$.
(D4) $(\forall q, r \in \mathbf{Q})[q < r \to q \in L \vee r \in R]$.
(D5) $(\forall q \in \mathbf{Q})\neg(q \in L \wedge q \in R)$.

(Axiom (D2) is actually superfluous, but we shall refer to it in Proposition 4.5.) For addition and multiplication of Dedekind cuts see [3]; the order is defined by $(L, R) < (L', R')$ iff $q \in L' \cap R$ for some $q \in \mathbf{Q}$. Denote by $\mathbb{R}_d$ the Dedekind real numbers of $\mathcal{N}$. These are indeed isomorphic to the Cauchy reals since dependent choice is valid in this topos (Theorem 3.4).

Let ${}^*\mathbb{Q}$ and ${}^*\mathbb{R}$ be the nonstandard extensions in $\mathcal{N}$ of $\mathbb{Q}$ and $\mathbb{R}$, respectively. We have $\mathbf{Q} \subseteq {}^*\mathbb{Q} \subseteq {}^*\mathbb{R}$. An element of $\xi \in {}^*\mathbb{R}$ is called *bounded* if for some $n \in \mathbf{N}$, $-n < \xi < n$; it is *infinitesimal* if for all $n \in \mathbf{N}$, $-2^{-n} < \xi < 2^{-n}$. Define for $\alpha, \beta \in {}^*\mathbb{R}$ the relation that α is *appreciably smaller than* β (symbolically $\alpha \prec \beta$) to hold iff there exists $n \in \mathbf{N}$ with $\alpha + 2^{-n} < \beta$. For $\xi \in {}^*\mathbb{R}$ define

$$L_\xi = \{q \in \mathbf{Q}\colon q \prec \xi\}, \quad R_\xi = \{r \in \mathbf{Q}\colon \xi \prec r\}.$$

Proposition 4.1. *In the internal logic of* $\mathcal{N}$, (L_ξ, R_ξ) *is a Dedekind cut for any bounded* $\xi \in {}^*\mathbb{R}$.

Proof. (D1) follows since ξ is bounded. The proofs for (D2)–(D5) are straightforward using the transfer principle. We illustrate with (D3) and (D4). As for (D3), suppose $q \in L$. Hence, $q + 2^{-n} < \xi$ for some $n \in \mathbf{N}$. Thus, $q + 2^{-n-1} + 2^{-n-1} < \xi$, and consequently $q + 2^{-n-1} \prec \xi$. The case for R is similar. To show (D4) assume $q < r$ for some $q, r \in \mathbf{Q}$. Then for some $n \in \mathbf{N}$, $q < q+2^{-n} < r-2^{-n} < r$. Hence, $q+2^{-n} < \xi$ or $\xi < r - 2^{-n}$ by transfer. Thus, $q \in L$ or $r \in R$. $\square$

Note that this proposition gives, internally to $\mathcal{N}$, a standard part map $\xi \mapsto (L_\xi, R_\xi)$ from the bounded elements of ${}^*\mathbb{R}$ to $\mathbb{R}_d$. The next result (Lemma 4.2) shows that the map is onto. We first remark that from axioms (D3) and (D4) it follows, by the internal logic of $\mathscr{N}$, that for any $n \in \mathbf{N}$ and any Dedekind cut (L,R) with $a \in L$, $b \in R$ that

$$(\exists c,d \in \mathbf{Q})[a < c \wedge d < b \wedge c \in L \wedge d \in R \wedge |d-c| < 2^{-n}]. \tag{8}$$

Now for any subsheaves L and R of $\mathbf{Q}$, $\mathscr{F} \Vdash$ "(L,R) is a Dedekind cut" implies by (D1) that there is an epimorphism $\mathscr{G} \twoheadrightarrow \mathscr{F}$ and two locally constant functions $a \in \mathbf{Q}(\mathscr{G})$, $b \in \mathbf{Q}(\mathscr{G})$ such that $\mathscr{G} \Vdash a \in L \wedge b \in R$. Given any such a,b and $n \in \mathbb{N}$ we get by (8) an epimorphism $\eta : \mathscr{H} \twoheadrightarrow \mathscr{G}$ and $c,d \in \mathbf{Q}(\mathscr{G})$ so that

$$\mathscr{H} \Vdash a\eta < c \wedge d < b\eta \wedge c \in L \wedge d \in R \wedge |d-c| < 2^{-n}.$$

Lemma 4.2. *In the internal logic of* $\mathscr{N}$: *For every Dedekind cut* (L,R) *there is a bounded* $\xi \in {}^*\mathbb{Q}$ *such that* $(L,R) = (L_\xi, R_\xi)$.

Proof. Suppose that $L,R \subseteq \mathbf{Q}$ are subsheaves and that $\mathscr{F}$ is a filter such that

$$\mathscr{F} \Vdash \text{"}(L,R) \text{ is a Dedekind cut"}. \tag{9}$$

From (9) and the remarks before the lemma follows that there is some $\varepsilon : \mathscr{F}_0 \twoheadrightarrow \mathscr{F}$ and $a_0, b_0 \in \mathbf{Q}(\mathscr{F})$ such that

$$\mathscr{F}_0 \Vdash a_0 \in L \wedge b_0 \in R \wedge |b_0 - a_0| < 1.$$

Suppose we have constructed $\mathscr{F}_n \overset{\varphi_{n-1}}{\twoheadrightarrow} \mathscr{F}_{n-1} \twoheadrightarrow \cdots \overset{\varphi_0}{\twoheadrightarrow} \mathscr{F}_0$ and $a_k, b_k \in \mathbf{Q}(\mathscr{F}_k)$ such that for $k = 0,\ldots,n$,

$$\mathscr{F}_k \Vdash a_k \in L \wedge b_k \in R \wedge |b_k - a_k| < 2^{-k} \tag{10}$$

and for $k = 0,\ldots,n-1$, $\mathscr{F}_{k+1} \Vdash a_k\varphi_k \le a_{k+1} \wedge b_{k+1} \le b_k\varphi_k$. From (9) and the remarks preceding the lemma follows

$$\mathscr{F}_n \Vdash (\exists a,b \in \mathbf{Q})[a_n < a \wedge b < b_n \wedge a \in L \wedge b \in R \wedge |b-a| < 2^{-(n+1)}]. \tag{11}$$

Hence, there exists $\varphi_n : \mathscr{F}_{n+1} \twoheadrightarrow \mathscr{F}_n$ and $a_{n+1}, b_{n+1} \in \mathbf{Q}(\mathscr{F}_{n+1})$ so that

$$\mathscr{F}_{n+1} \Vdash a_n\varphi_n < a_{n+1} \wedge b_{n+1} < b_n\varphi_n \wedge a_{n+1} \in L \wedge b_{n+1} \in R \wedge |b_{n+1} - a_{n+1}| < 2^{-(n+1)}.$$

Now, construct $\mathscr{G}$ and $\gamma_n : \mathscr{G} \twoheadrightarrow \mathscr{F}_n$ exactly as in the proof of Lemma 3.3, so that $\varphi_n\gamma_{n+1} = \gamma_n$. Define $\alpha, \beta \in {}^*\mathbb{Q}(\mathscr{G})$ by $\alpha(n,x) = a_n(x)$ and $\beta(n,x) = b_n(x)$. It follows using Theorem 2.1 that for each $n \in \mathbb{N}$,

$$\mathscr{G} \Vdash |\beta - \alpha| < 2^{-n}, \tag{12}$$

and moreover that

$$\mathscr{G} \Vdash a_n\gamma_n \le \alpha \wedge \beta \le b_n\gamma_n. \tag{13}$$

We shall prove that

$$\mathscr{G} \Vdash (\forall q \in \mathbf{Q})(q \in L \leftrightarrow q \prec \alpha), \tag{14}$$

$$\mathscr{G} \Vdash (\forall q \in \mathbf{Q})(q \in R \leftrightarrow \alpha \prec q). \tag{15}$$

Since $\varepsilon\gamma_0 : \mathscr{G} \to \mathscr{F}$ is epi, (14) and (15) with $\xi = \alpha$ give the desired conclusion

$$\mathscr{F} \Vdash \text{“}(L,R) = (L_\xi, R_\xi)\text{”}.$$

We show only (14) since (15) is completely analogous using (12). Let $\tilde{a}_n = a_n\gamma_n$ and $\tilde{b}_n = b_n\gamma_n$. Suppose $\mathscr{H} \Vdash \theta \in L$, where $\theta \in \mathbf{Q}(\mathscr{H})$ and $\eta : \mathscr{H} \to \mathscr{G}$. By (D3) $\mathscr{H} \Vdash (\exists n \in \mathbf{N})\theta + 2^{-n} \in L$. Hence, there are $n_1, \dots, n_k \in \mathbb{N}$ and a cover $\{\eta_i : \mathscr{H}_i \to \mathscr{H}\}_{i=1}^k$ such that $\mathscr{H}_i \Vdash \theta\eta_i + 2^{-n_i} \in L$ for all $i = 1, \dots, k$. Let $n = \max(n_1, \dots, n_k)$ and we have by (D2), $\mathscr{H}_i \Vdash \theta\eta_i + 2^{-n} \in L$. Thus, since $\{\eta_i\}$ is a cover,

$$\mathscr{H} \Vdash \theta + 2^{-n} \in L.$$

From (10) follows that $\mathscr{H} \Vdash \theta < \tilde{a}_n\eta$, and hence $\mathscr{H} \Vdash \theta \prec \tilde{a}_n\eta$. Then by (13) $\mathscr{H} \Vdash \theta \prec \alpha\eta$. This proves the direction ($\Rightarrow$). To prove ($\Leftarrow$) assume $\mathscr{H} \Vdash \theta \prec \alpha\eta$ for $\theta \in \mathbf{Q}(\mathscr{H})$ and $\eta : \mathscr{H} \to \mathscr{G}$. Hence, as above, there is some $n \in \mathbb{N}$, $\mathscr{H} \Vdash \theta + 2^{-n} < \alpha\eta$. Since $\mathscr{H} \Vdash |\tilde{a}_n\eta - \alpha\eta| \leq 2^{-n}$, we have $\mathscr{H} \Vdash \theta \leq a_n\eta$ and so $\mathscr{H} \Vdash \theta \in L$. □

The set of bounded elements of ${}^*\mathbb{Q}$ is denoted by $F = \mathrm{Fin}({}^*\mathbb{Q})$, and the set of infinitesimal elements is called I. The operations $+$ and $\cdot$ and the order $\prec$ extend to the quotient ring F/I. Then $\phi : F/I \to \mathbb{R}_d$ given by $\phi([\xi]) = (L_\xi, R_\xi)$ is well defined.

Theorem 4.3. *In the internal logic of $\mathscr{N}$ the map $\phi : F/I \to \mathbb{R}_d$ is an isomorphism of ordered rings.*

Proof. Note that if $L_\xi = \{q \in \mathbf{Q}\colon q < 0\}$ and $R_\xi = \{q \in \mathbf{Q}\colon 0 < q\}$, then ξ is infinitesimal. Hence ϕ is mono. Lemma 4.2 shows that ϕ is epi. Moreover, $\xi \prec \eta$ iff for $\xi \prec q \prec \eta$ some $q \in \mathbf{Q}$, i.e., $(L_\xi, R_\xi) < (L_\eta, R_\eta)$. It is a straightforward but tedious exercise to check that ϕ is a homomorphism with respect to addition and multiplication. □

Apart from the standard properties of Dedekind reals in any topos, see [3, Theorem 6.65], $\mathbb{R}_d$ has some classical features within $\mathscr{N}$.

Corollary 4.4. *In the internal logic of $\mathscr{N}$ we have*
(i) *$\mathbb{R}_d$ is an integral domain.*
(ii) *For all $x, y \in \mathbb{R}_d$: $x \leq y$ or $y \leq x$.*
(iii) *The intermediate value theorem holds for $\mathbb{R}_d$.*

Proof. Parts (i) and (ii) follow from Propositions 34 and 32 in [8] using the isomorphism of the theorem. By (ii) and Theorem 3.4 we can use the classical bisection method to prove (iii). □

Given a Dedekind cut (L, R) in $\mathscr{N}$ one may ask what information about concrete rational approximations can in general be extracted. (See [10] for an analogous problem.) For each filter $\mathscr{F}$ define

$$L_{\mathscr{F}} = \{q \in \mathbb{Q}\colon {}^*q \in L(\mathscr{F})\}, \qquad R_{\mathscr{F}} = \{q \in \mathbb{Q}\colon {}^*q \in R(\mathscr{F})\}.$$

Proposition 4.5. *Let* (L,R) *be a Dedekind cut in* $\mathscr{N}$. *For any filter* $\mathscr{F}$, *then* $(L_{\mathscr{F}},R_{\mathscr{F}})$ *satisfies axioms* (D1)–(D3). *Moreover*,

(i) *if* $\mathscr{F}$ *is proper*, *then the consistency axiom* (D5) *is also satisfied*,
(ii) *and if* $\mathscr{F}$ *is an ultrafilter*, *then* $(L_{\mathscr{F}},R_{\mathscr{F}})$ *is a Dedekind cut*.

Proof. Let $\mathscr{F}$ be an arbitrary filter. We prove (D1)−(D3) for $(L_{\mathscr{F}},R_{\mathscr{F}})$ assuming (D1)−(D5) for (L,R). (D1): We have $\mathscr{F} \Vdash (\exists q \in \mathbf{Q})q \in L$, so there is a cover $\{\varphi_i : \mathscr{F}_i \to \mathscr{F}\}_{i=1}^k$ and $q_1,\dots,q_k \in \mathbb{Q}$ such that $\mathscr{F}_i \Vdash {}^*q_i \in L$. Hence, by (D2), $\mathscr{F} \Vdash {}^*q \in L$ for $q = \min(q_1,\dots,q_k)$. The case for R is analogous. (D2): Immediate. (D3): Suppose $\mathscr{F} \Vdash {}^*q \in L$. By (D3), $\mathscr{F} \Vdash (\exists r \in \mathbf{Q})(r > {}^*q \wedge r \in L)$. Hence, there is a cover $\{\varphi_i : \mathscr{F}_i \to \mathscr{F}\}_{i=1}^k$ and $r_1,\dots,r_k \in \mathbb{Q}$ such that $\mathscr{F}_i \Vdash {}^*r_i > {}^*q \wedge {}^*r_i \in L$. If $r_i \le q$ for all $i = 1,\dots,k$, then $\mathscr{F}$ is non-proper so this case is trivial. Otherwise, let $r = \min\{r_i\colon r_i > q\}$. Then $\mathscr{F}_i \Vdash {}^*r > {}^*q \wedge {}^*r \in L$ (since if $r_i \le q$, then $\mathscr{F}_i$ is non-proper). Hence, $r \in L(\mathscr{F})$ and $r > q$. The case for R is similar.

Now suppose that $\mathscr{F}$ is proper. Now, if $\mathscr{F} \Vdash {}^*q \in L \wedge {}^*q \in R$, then by (D5), $\mathscr{F} \Vdash \perp$. But since $\mathscr{F}$ is proper, this is a contradiction. Hence, (D5) holds for $(L_{\mathscr{F}},R_{\mathscr{F}})$.

If $q < r$, then $\mathscr{F} \Vdash {}^*q \in L \vee {}^*r \in R$ by (D4). In case $\mathscr{F}$ is an ultrafilter, $\mathscr{F} \Vdash {}^*q \in L$ or $\mathscr{F} \Vdash {}^*r \in R$ (see [8, Corollary 51]). This last step is non-constructive. □

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References

[1] E. Bishop, D.S. Bridges, Constructive Analysis, Springer, Berlin, 1985.
[2] A. Blass, Two closed categories of filters, Fund. Math. 94 (1977) 129–143.
[3] P.T. Johnstone, Topos Theory, Academic Press, New York, 1977.
[4] V. Koubek, J. Reiterman, On the category of filters, Comm. Math. Univ. Carolinea 11 (1970) 19–29.
[5] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic, Springer, Berlin, 1992.
[6] I. Moerdijk, A model for intuitionistic nonstandard arithmetic, Ann. Pure Appl. Logic 73 (1995) 37–51.
[7] E. Palmgren, Sheaf-theoretic foundation for nonstandard analysis, Ann. Pure Appl. Logic 85 (1997) 69–86.
[8] E. Palmgren, Developments in constructive nonstandard analysis, Bull. Symbolic Logic 4 (1998) 233–272.
[9] E. Palmgren, Constructive nonstandard representations of generalized functions, Indag. Math. N.S. 11 (2000) 129–138.
[10] P. Schuster, A constructive look at generalized Cauchy reals, Math. Logic Quarterly 46 (2000) 125–134.
[11] K.D. Stroyan, W.A.J. Luxemburg, Introduction to the Theory of Infinitesimals, Academic Press, New York, 1976.